

## INTRODUCING $q$ -DEFORMATION ON THE LEVEL OF VECTOR VARIABLES

F. Sommen\* and M. Watkins\*\*

### Abstract

In this paper we investigate  $q$ -deformation on the level of vector variables instead of coordinates.

### Introduction

By Clifford polynomial algebra we understand the algebra generated by a set of  $m$  real variables  $\{x_1, \dots, x_m\}$  together with a set of basis elements  $\{e_1, \dots, e_m\}$  generating a Clifford algebra with defining relations  $e_i e_j + e_j e_i = 2g_{ij}$ . Within this algebra one may define vector variables  $\underline{x} = \sum_{i=1}^m x_i e_i$  and one may consider endomorphisms on this algebra such as the Dirac operator or vector derivative  $\partial_{\underline{x}}$  (see also [4]). In [3] a geometric calculus has been developed in which the calculus in Clifford algebras was formulated in a coordinate independent way. The vectors in this algebra are supposed to belong to some Clifford algebra of infinite (or unbounded) dimension, called geometric algebra. This point of view has lead to our paper [5] in which we introduced an algebra of abstract vector variables as follows. Let  $S$  be a set of so-called abstract vector variables and assume that the anti-commutator  $\{x, y\} = xy + yx$  is a commutative scalar, i.e.

$$(A1) \quad [z, \{x, y\}] = 0, \quad x, y, z \in S.$$

Then the “radial algebra”  $R(S)$  is defined as the universal algebra generated by the set  $S$ , taking (A1) into account. We have shown in [5] that, in case  $S$  is finite, the algebra  $R(S)$  may be thought of as an algebra of Clifford-vector variables of the above type  $\underline{x} = \sum_{j=1}^m x_j e_j$ . But the dimension  $m$  depends on the cardinality of  $S$ . Hence a Clifford-vector representation of any  $R(S)$  is only possible in an infinite dimensional Clifford algebra.

One could hence say that any algebra  $R(S)$  is a subalgebra of an infinite (not finite) dimensional geometric algebra. To avoid the complications with infinite dimensionality (which do arise when defining Dirac operators) we developed the theory in a purely abstract way, not using any Clifford algebra representation.

The next question is: can Clifford algebra be linked up with the recent developments in non-commutative geometry ( $q$ -deformations). In [6] we presented a

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\* Senior Research Associate, NFWO, Belgium

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“ $q$ -deformed version of Clifford analysis” based on the use of the commutation relations  $x_i x_j = q_{ij} x_j x_i$  for coordinates and of the “quantum Clifford algebras” introduced in [1]. But in fact it turned out that we defined a representation of a radial algebra  $R(S)$  on phase space, using non-commutative coordinates instead of commuting ones. As a result, the zonal functions are the same as in the commutative case; there is no real  $q$ -deformation involved in this part of the function theory.

To obtain new zonal functions, one practically has to construct a  $q$ -deformed version of the algebra of vector variables itself. This is what we present in section one. However, it turns out that with the aid of extra non-commuting scalar variables, a  $q$ -deformed algebra of vector variables can always be transformed back to a usual one (see [5]). The same trick applies to the so-called “quantum Clifford algebras” introduced in [1]; these algebras are not really different from the standard Clifford algebras.

One may thus wonder whether anything like Clifford algebra can be linked to non-commutative geometry at all. In section two we illustrate that the calculus in noncommutative coordinates presented in [2] may lead to a  $q$ -deformed algebra of vector variables on phase space. It shows that the zonal functions which occur in noncommutative coordinates may indeed be incorporated in the algebra of vector variables. Yet the interpretation of abstract vector variables in terms of Clifford algebras seems to conflict with the  $R$ -matrix formalism. A generalisation of Clifford algebra is needed to express the  $q$ -deformed algebra of vector variables in coordinates. Eventually this leads to an algebra of four vector variables with constraints which can no longer be transformed back to a usual one.

## 1. $q$ -Deformed Versions of the Algebra of Abstract Vector Variables

Let us again start from a set  $S$  of abstract vector variables; then the axiom (A1) indicates that the anti-commutator  $xy + yx$  is a scalar. We adapt this axiom in several steps. First we state

**Definition 1.** The  $q$ -deformed anti-commutator of two vector variables is given by  $\{x, y\} = xy + qyx$ .

In this definition it is understood that the constant  $q = q_{xy}$  depends on the choice of the pair of vector variables  $(x, y) \in S \times S$  and that

$$q_{xy}q_{yx} = 1, \quad q_{xx} = 1$$

so that the anti-commutator satisfies the relation

$$\{x, y\} = q_{xy}\{y, x\}.$$

Assumption (A1) could now be restated as

$$(A1) \quad \{x, y\} \text{ is a scalar}$$

This is fair enough, but the problem is that the word “scalar” may have to be given a more general meaning. It is still accepted that sums and products of elements with the label “scalar” inherit this label, but scalars may no longer commute either among themselves or with vectors. Due to the definition of  $\{x, y\}$  in terms of vector variables, the commutation relations among the “scalars”  $\{x, y\}$  are determined by intertwining relations for  $\{x, y\}$  with any vector  $z$ . The simplest idea would be to assume that

$$(A2) \quad z\{x, y\} = p\{x, y\}z$$

where again  $p = p_{xyz}$  depends on the choice of the triple  $(x, y, z)$  and has to satisfy the relation  $p_{xyz} = p_{yxz}$ . But this still does not quite determine what are scalars and what are vectors. To that end we assume an extended form of linear independence of vectors, knowing that any finite set of different vector variables  $x_1, \dots, x_k$  is already supposed to be linearly independent over the real numbers.

Let  $x_1, \dots, x_k \in S$  be  $k$  different vector variables and consider objects  $s_1, \dots, s_k$  which carry the label “scalar”, then

$$(A3) \quad \sum_{j=1}^m s_j x_j = 0 \text{ implies } s_j = 0.$$

This determines a relation between scalars and vectors. It leads to:

**Theorem 1.** The constants  $p_{xyz}$  are given by  $p_{xyz} = q_{zx}q_{zy}$ .

**Proof.** Direct application of Definition 1 leads to

$$\begin{aligned} z\{x, y\} &= zxy + q_{xy}zyx \\ &= \{z, x\}y + q_{xy}\{z, y\}x - q_{zx}xzy - q_{xy}q_{zy}yzx \\ &= \{z, x\}y + q_{xy}\{z, y\}x - q_{zx}x\{z, y\} - q_{xy}q_{zy}y\{z, x\} + q_{zx}q_{zy}\{x, y\}z. \end{aligned}$$

This identity is satisfied under the condition  $p_{xyz} = q_{zx}q_{zy}$  and, due to (A2) and (A3), this condition is also necessary.  $\square$

This theorem however immediately leads to a possibility of reducing the above defined algebra to a usual algebra of vector variables as follows. For each  $x \in S$  we introduce a new coordinate  $l_x$  and we assume the commutation relations

$$l_y l_x = q_{xy} l_x l_y.$$

If we then introduce the new vector variables  $X = l_x x$ ,  $x \in S$ , we find that

$$\{X, Y\} = XY + YX = l_x l_y (xy + q_{xy} yx)$$

and also that

$$Z\{X, Y\} = l_z l_x l_y z(xy + q_{xy} yx) = \{X, Y\}Z.$$

Hence, freely generated  $q$ -deformed algebras of vector variables can always be rescaled back to non  $q$ -deformed ones.

## 2. Comparison with Calculus in the Quantum Plane

We consider coordinates  $x^i$  and derivatives  $\partial_i = \partial_{x^i}$  for which we have the commutation relations

$$\begin{aligned} x^i x^j &= q^{-1} R_{kl}^{ij} x^k x^l, \quad \partial_l \partial_k = q^{-1} R_{kl}^{ij} \partial_j \partial_i \\ \partial_k x^i &= \delta_k^i + q R_{kl}^{ij} x^l \partial_j. \end{aligned}$$

Hereby  $R$  is an invertible solution of the quantum Yang-Baxter equation satisfying the characteristic equation

$$R^2 - (q - q^{-1})R - 1 = 0.$$

Moreover one can consider another set of coordinates  $a^i$  which satisfy commutation relations similar to the  $x$ -coordinates and for which we have the intertwining relations

$$x^i a^j = q R_{kl}^{ij} a^k x^l.$$

Using the characteristic equation one then shows that  $y^i = x^i + a^i$  still satisfy the same commutation relations as the  $x$ - and the  $a$ -coordinates. In this way translation in the quantum plane is defined. In [2] the authors also consider the linear momentum coordinates  $p_k$  satisfying the appropriate commutation and intertwining relations. The authors then define something like a dot-product

$$a \cdot \partial = \sum_{i=1}^m a^i \partial_i$$

and in this way they also consider  $x \cdot \partial$ ,  $x \cdot p$ ,  $a \cdot p$ , arriving at commutation relations of the following form

$$\begin{aligned}(a \cdot d)d_i &= q^2 d_i(a \cdot d) \quad , \quad (a \cdot d)a^i = q^{-2} a^i(a \cdot d) \\ (a \cdot d)x^i &= x^i(a \cdot d) \quad , \quad (x \cdot d)x^i = q^2 x^i(x \cdot d) \\ (x \cdot d)a^i &= a^i(x \cdot d) \quad , \quad d_i(x \cdot d) = q^2(x \cdot d)d_i \\ (x \cdot p)(a \cdot p) &= q^2(a \cdot p)(x \cdot p)\end{aligned}$$

where  $p_i$  commutes with both  $x \cdot d$  and  $a \cdot d$ . Here  $d_i$  is the same as  $\partial_i$  except that we forget the action of  $\partial_i$  on the  $x$ -coordinates. To be workable we rewrite these identities. We formally introduce vector notation  $x$  for  $x^i$ ,  $a$  for  $a^i$ , etc. and we put  $b \cdot c = \frac{1}{2}\{b, c\}$ . Then we do actually get purely symbolic identities of the form

$$\begin{aligned}\{a, d\}d &= q^2 d\{a, d\} \quad , \quad \{a, d\}a = q^{-2} a\{a, d\} \\ \{a, d\}x &= x\{a, d\} \quad , \quad \{x, d\}x = q^2 x\{x, d\} \\ \{x, d\}a &= a\{x, d\} \quad , \quad d\{x, d\} = q^2\{x, d\}d \\ \{x, d\}p &= p\{x, d\} \quad , \quad \{a, d\}p = p\{a, d\} \\ \{x, p\}\{a, p\} &= q^2\{a, p\}\{x, p\}.\end{aligned}$$

The next question is to interpret these relations in terms of radial algebra. There actually is a solution to this, namely to generate the algebra from  $S = \{x, a, d, p\}$  with constraints

$$\begin{aligned}x^2 &= a^2 = d^2 = p^2 = 0 \\ xa + q^2 ax &= 0 \quad , \quad pd + q^2 dp = 0,\end{aligned}$$

and where the commutator  $\{, \}$  is given by

$$\begin{aligned}\{x, d\} &= xd + q^{-2} dx \quad , \quad \{x, p\} = xp + q^2 px \\ \{a, d\} &= ad + q^2 da \quad , \quad \{a, p\} = ap + q^2 pa.\end{aligned}$$

Hence, as long as one considers zonal functions (functions depending on dot products like  $x \cdot p$  or  $a \cdot p$ ) and zonal operators, there seems to be a way of introducing  $q$ -deformed abstract vector variables.

But the problem is that, if one writes  $x = x^i f_i$ ,  $a = a^i f_i$  and so on, thus introducing real vectors (we assume the  $x^i$  and  $f_i$  commute), then the relation  $xa + q^2 ax = 0$  leads to the generalised Clifford algebra defining relations

$$f_i f_j = -q R_{ij}^{kl} f_k f_l$$

from which it follows that  $x^2 = a^2 = (x + a)^2 = 0$ , so that  $xa + ax = 0$  or  $q^2 = 1$ . The amazing thing is that, on the abstract level the above relations

seem to work well. But of course one then restricts one's attention to the elements belonging to the algebra generated by the set  $\{x, a, p\}$ .

The only idea which seems to work however is simply to choose a new Clifford algebra basis for each variable in the theory. This would here correspond to something like

$$x = x^j X_j, \quad a = a^j A_j, \quad d = d_j D^j, \quad p = p_j P^j,$$

where the elements  $X_j, A_j, D^j, P^j$  are supposed to generate something like a Clifford algebra. The commutation relations for the  $X_j$  are determined by requiring that, whenever  $y^j$  are coordinates with commutation relations  $y^i y^j = q^{-1} R_{kl}^{ij} y^k y^l$ , then the square of  $y^j X_j$  vanishes. In particular for  $y^j = x^j + a^j$  this leads to  $x^i a^j X_i X_j = -a^k x^l X_k X_l$ , from which we get

$$X_k X_l = -q R_{kl}^{ij} X_i X_j.$$

A similar relation holds for the generators  $A_j, D^j, P^j$ . Next from the relation  $xa + q^2 ax = 0$  one obtains the intertwining relations

$$A_k X_l = -q^{-1} R_{kl}^{ij} X_i A_j.$$

and a similar intertwining relation holds for  $P^j, D^j$ . From the relation  $xd + q^{-2} dx = \{x, d\} = x^i d_i$  one obtains that  $x^l d_j X_l D^j + q^{-2} d_k x^i D^k X_i$  which leads to

$$X_l D^j + q^{-1} R_{kl}^{ij} D^k X_i = \delta_l^j.$$

Similar intertwining relations exist between  $X_j, P^j$ , between  $A_j, D^j$  and between  $A_j, P^j$ . All these relations together determine an algebra which in case  $q = 1$ ,  $X_j = A_j = f_j$ ,  $D^j = P^j = f_j'$  would correspond to the Clifford algebra  $R_{m,m}$  on phase space. But in the  $q$ -deformed setting, this Clifford algebra seems to "split up into copies", depending on the number of variables.

At the present time we can reintroduce the differential operator  $\partial_x = \partial_{x_j} D^j$  instead of the variable  $d$  which only played a formal role as a symbol of  $\partial_x$ . The operator  $\partial_x$  is a mixture between an exterior derivative and a Dirac operator and may be compared with the affine Dirac operators considered in [5].

Hence in our analysis of the paper [2] only the variables

$$x = x^j X_j, \quad a = a^j A_j, \quad p = p_j P^j$$

play a role as true vector variables. The relations which determine the radial algebra generated by  $x, a, p$  are

$$\begin{aligned} x^2 = a^2 = p^2 = 0, \quad xa + q^2 ax = 0, \\ xp + q^2 px = \{x, p\}, \quad ap + q^2 pa = \{a, p\}. \end{aligned}$$

As in the previous section we may now consider the scale transformation

$$X = l_x x, \quad A = l_a a, \quad P = l_p p,$$

where

$$q^2 l_x l_a = l_a l_x, \quad q^2 l_x l_p = l_p l_x, \quad q^2 l_a l_p = l_p l_a$$

and we arrive at the “rectified defining relations”

$$\begin{aligned} X^2 = A^2 = P^2 = 0, \quad XA + AX = 0 \\ XP + PX = \{X, P\}, \quad AP + PA = \{A, P\}, \end{aligned}$$

where  $\{X, P\}$  and  $\{A, P\}$  are commutative scalars. Hence it seems as if nothing is the matter and that the calculus presented in [2] may in principle be transformed into a calculus in commutative coordinates. Indeed, the vector variables  $X, A, P$  are representable by usual vector variables in a Clifford algebra. But in fact the variables  $\{x, a, p\}$  do not take the characteristic equation of the  $R$ -matrix into account. On considering [2] we see that this equation plays a crucial role in defining the translation  $x^j \rightarrow x^j + a^j$ . We used this translation to obtain the defining relations for the algebra generated by the elements  $X_j$ . Hence this algebra cannot be defined by merely considering the variables  $x, a, p$  and their algebraic behaviour; we need to consider the map  $x^j \rightarrow x^j + a^j$ .

Of course, looking to the above representation of the variables  $x$  and  $a$ , this translation map is the map  $x^j X_j \rightarrow (x^j + a^j) X_j$  which produces the “new vector variable”  $a_x = a^j X_j$  which satisfied  $a_x^2 = 0$  and which is linked to the vector variables  $x, a, p$  by the relations

$$\begin{aligned} xa_x + a_x x = 0, \quad aa_x + a_x a = 0 \\ a_x p + q^2 pa_x = \sum_j a_j p_j = ap + q^2 pa. \end{aligned}$$

We hence have a radial algebra with constraints generated by the variables  $\{x, a_x, a, p\}$ .

Now we can illustrate an interesting phenomena. Knowing that any freely generated  $q$ -deformed radial algebra can be “rectified into a normal radial algebra” and knowing also that this rectification works for the algebra generated

by  $\{x, a, p\}$  we can now try to extend it to the algebra generated by  $\{x, a_x, a, p\}$  by considering a transformation of the form  $A_x = l_{a_x} a_x$ . From the constraint  $a_x p + q^2 p a_x = ap + q^2 pa$  we obtain the relation

$$AP + PA = l_a a_x P + P l_a a_x$$

which forces us to put  $A_x = l_a a_x$ , i.e.  $l_a = l_{a_x}$ . One may refuse to make this choice, but then the extra  $l$ -parameters will not disappear and the transformed algebra cannot be a normal radial algebra which is representable by real vector variables in a Clifford algebra. The  $l$ -parameters are only useful for the sake of transformation and must ultimately disappear. But when taking  $l_{a_x} = l_a$ , we obtain that the “rectified relation”  $xa_x + a_x x = 0$  transforms into the “ $q$ -deformed relation”

$$XA_x + q^{-2}A_x X = 0.$$

Hence the transformation  $x \rightarrow X$ ,  $a \rightarrow A$ ,  $a_x \rightarrow A_x$ ,  $p \rightarrow P$  removes the  $q$ -parameter at one place and installs it at another, as if it were an air bubble under wallpaper. Hence,  $q$ -deformed radial algebras with constraints cannot be rectified in the same way as freely generated algebras; they have “quantum bubbles”.

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